COMPACTA IN THE STONE-ČECH REMAINDER OF R^n

BY

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In this dissertation, new techniques for constructing continua in the Stone-Čech remainder of a completely regular Hausdorff space are illustrated for the special case of n-dimensional Euclidean space, where $n \geq 2$. These techniques depend heavily on results obtained in the area of shape theory and appear to be applicable to more general spaces as well as to R^n. After establishing basic concepts and terminology relating to the Stone-Čech compactification, shape theory, and Čech cohomology, some preliminary lemmas are proven which are then used in the proofs of subsequent theorems. These lemmas illustrate the distinctive properties of the Stone-Čech compactification. After these preliminaries, c continua are constructed in $\beta R^3 - R^3$ such that no two have isomorphic first Čech cohomology groups and, hence, are not of the same shape. Then, a particular

embedding of subsets of the continua into βR^3 will exhibit 2^c compacta in $\beta R^3 - R^3$, no two of which have the same shape. An easy modification of the compacta yields 2^c nonhomeomorphic continua in $\beta R^3 - R^3$, the proof of which utilizes the properties of shape dimension. It is shown that these results for βR^3 imply analogous results for βR^n, where $n \geq 3$. The case of $n = 2$ is treated in the subsequent chapter, where 2^c nonhomeomorphic continua are constructed in the Stone-Čech remainder of R^2.

CHAPTER ONE
INTRODUCTION

Shape theory is a generalization of homotopy theory that agrees with homotopy theory in the case of CW complexes. For more general spaces, however, shape theory reflects better the global behavior of the space. For example, if X is the graph of the sin (1/x) curve closed by an arc, commonly referred to as the "Warsaw circle," then X has the homotopy type of a point, yet is shape equivalent to a circle. Roughly, this follows from the fact that X is the intersection of neighborhoods in R^2 that are homotopy equivalent to S^1.

In this dissertation, we will be concerned with the application of shape theory to the Stone-Čech compactification of a space X, βX, and in particular, to βR^n for $n \geq 2$. Previously, much work has been done in the study of the Stone-Čech compactification of the natural numbers, βN, that does not involve shape theory. Until recently, a major method used to study βX for a more general space X has been to embed a copy of the natural numbers into X and apply results already known for βN to βX. While this method has obvious restrictions, shape theory has offered new methods for examining βX and $\beta X - X$ that utilize the intrinsic topological properties of the space X, as will be illustrated in this text in the case of βR^n.

A basic tool used in the area of shape theory is the computation of Čech cohomology groups. Just as the singular cohomology groups are invariants in the homotopy category, so are the Čech cohomology groups invariants in the shape category. In [1], A. Calder and J. Siegel have proven results that enable us to easily compute the Čech cohomology groups of certain types of continua contained in βR^n.

We will begin by establishing in Chapter Two the necessary background material required to understand the basic concepts and terminology that will be used throughout the succeeding chapters, namely, concepts related to the Stone-Čech compactification, shape theory, and Čech cohomology. Then, in Chapter Three, we will establish some preliminary lemmas which will be used in the proofs of the subsequent theorems. The proofs of these lemmas will illustrate the distinctive properties of the Stone-Čech compactification.

We begin Chapter Four by constructing c continua in $\beta R^3 - R^3$ which have nonisomorphic first Čech cohomology groups. First, we will form a specific collection of solenoids that has cardinality c. Then, corresponding to each solenoid in this collection, we will construct a continuum in $\beta R^3 - R^3$ by attaching an infinite number of copies of the solenoid end-to-end in R^3 along the x-axis, and considering the Stone-Čech remainder of this set. This will be a continuum in $\beta R^3 - R^3$ whose first Čech cohomology group will have a distinctive divisibility property that will distinguish it from the other first Čech cohomology groups of continua formed in a like manner from different solenoids in the collection. This construction yields c continua in $\beta R^3 - R^3$ with different shapes.

Next, in Theorem 4.2, a particular embedding of subsets of these continua into βR^3 will exhibit 2^c compacta in $\beta R^3 - R^3$, no two of which have the same shape. More specifically, for each collection of continua of cardinality c, we will embed the collection in $\beta R^3 - R^3$ and consider the closure of the union of the continua. This set will be a compactum in $\beta R^3 - R^3$ that contains each of the continua as an isolated component. Then, any two such compacta will have different shapes, since any shaping between them induces a shaping between the isolated components, i.e., between the original continua. Thus, we will have obtained a contradiction to Theorem 4.1.

Finally, in Theorem 4.3, an easy modification of the compacta constructed in the proof of Theorem 4.2 will yield 2^c nonhomeomorphic continua in $\beta R^3 - R^3$, the proof of which utilizes the properties of shape dimension as developed by J. Keesling in [9]. The proof involves taking each compactum in Theorem 4.2 and adding to it the Stone-Čech remainder of a plane. The resulting set will be a continuum contained in $\beta R^3 - R^3$. It will then be shown that any homeomorphism between two such continua must take the original compacta onto each other, implying that the compacta are homeomorphic. This will contradict Theorem

4.2, since two spaces that are not shape equivalent cannot be homeomorphic.

Since R^3 is a closed subset of R^n for $n \geq 3$, the above results also hold for βR^n, where $n \geq 3$.

In Chapter Five, we prove that $\beta R^2 - R^2$ also contains 2^c nonhomeomorphic subcontinua. While this implies the result for $n \geq 3$, the construction in dimension three yields the additional shape-theoretic results that there are c continua in $\beta R^3 - R^3$ with nonisomorphic first Čech cohomology groups and 2^c compacta with different shapes. Moreover, the construction in R^3 may be useful for obtaining additional shape-theoretic results than those given here. In particular, it seems reasonable that the continua constructed in $\beta R^3 - R^3$ may be shown to have different shapes or even nonisomorphic first Čech cohomology groups. In the case of $\beta R^2 - R^2$, it seems unlikely that any additional shape-theoretic results can be obtained with this construction. The case of R^2 is intentionally treated after Chapter Four, since an understanding of the constructions in Theorems 4.1 and 4.2 will aid the reader in the understanding of the proof of Theorem 5.1.

The number of nonhomeomorphic continua contained in $\beta R^1 - R^1$ is yet unknown. Since the only unbounded connected subsets of R^1 are rays, the methods used in Chapters Four and Five to construct continua in $\beta R^n - R^n$ for $n \geq 2$ cannot be applied in this case.

The survey articles [10], [13], and [15] provide further information on recent advances in shape theory.

CHAPTER TWO
BACKGROUND MATERIAL

In this chapter, we will establish the basic definitions, concepts, and notation that will be used throughout the remainder of this text. The material found in this chapter provides the reader with the background necessary for the understanding of succeeding chapters.

The Stone-Čech Compactification

A compactification of a topological space X is a compact Hausdorff space K, together with an embedding e: X → K such that e(X) is dense in K. Thus, we may consider X as a subspace of K by identifying X with e(X). The Stone-Čech compactification of X, denoted by βX, is a compactification of X with the property that every bounded, real continuous function defined on X has a continuous extension to all of βX.

For a topological space X, let C(X) denote the ring of all real-valued, continuous functions defined on X, and C*(X) the subring of all bounded functions in C(X). A subspace Y of X is said to be C-embedded (respectively, C*-embedded) in X if every element of C(Y) (respectively, C*(Y)) has an extension in C(X) (respectively, C*(X)) to all of X. Hence, the Stone-Čech compactification of X is a compact Hausdorff space containing X as a dense, C*-embedded subspace.

The question arises as to which spaces admit a Stone-Čech compactification. These are precisely the completely regular (Hausdorff) spaces. Necessity is clear since every subspace of a compact space is completely regular. We now construct the Stone-Čech compactification for a completely regular space X. For every $f \in C^*(X)$, let I_f denote the range of f. Since f is bounded, the closure of I_f, $cl(I_f)$, is compact. Consider the mapping $e\colon X \to \Pi cl(I_f)$, where $e(x)_f = f(x)$, and let $\beta X = cl(e(X))$. The function e is continuous, since its composition with each projection is continuous ($\pi_f \circ e = f$). To show e is an embedding, let U be an open set in X and $x \in U$. Let $f \in C^*(X)$ such that $f(x) \notin cl(f(X - U))$. (This is possible by complete regularity.) Then the set $\{y \in e(X)\colon y_f \notin cl(f(X - U))\}$ is a neighborhood of e(X) and is contained in f(U). Hence, f(U) is open in e(X). Then, since e is clearly one-to-one, e is an embedding. Note that if X is compact, then e(X) is closed in $\Pi cl(I_f)$, and hence, βX is homeomorphic to X. Finally, let $h\colon X \to R$ be a bounded continuous, real-valued function, and let $I = cl_R(h(X))$. We will show that h has a continuous extension to a map from βX to I. Since I is compact, I is homeomorphic to βI. Therefore, we may consider h as a map from X into βI. Now, define $\beta h\colon \beta X \to \beta I$ by

$$(\beta h(\{t_f\}))_g = t_{g \circ h}$$

Then, $(\beta h \circ e(x))_g = (\beta h(\{e(x)_f\}))_g$

$$= e(x)_{g \circ h}$$

$$= (g \circ h)(x)$$

$$= h(x)_g.$$

Hence, $\beta h \circ e = h$, and so βh extends h. Therefore, βX is the Stone-Čech compactification of X as defined earlier. Note that this construction yields the algebraic result that $C^*(X)$ is isomorphic to $C^*(\beta X) = C(\beta X)$.

From now on, all spaces are assumed to be completely regular.

The Stone-Čech compactification has the following well-known properties due to M. Stone and E. Čech. For every compact space Y and continuous function $f\colon X \to Y$, there exists a unique continuous map $F\colon \beta X \to Y$ such that $f = F \circ i$, where i is the inclusion map from X into βX. In addition, any compactification of X that also has this property is homeomorphic to βX via a homeomorphism which is the identity on X. Also, βX is the "largest" compactification of X in that every other compactification of X is a quotient space of βX (see [3]).

For a topological space X, let X^* denote the subspace $\beta X - X$, called the Stone-Čech remainder of X. In general, X^* is fairly complicated. For example, $\beta R^n - R^n$ contains no nondegenerate metric continua (see [11]), yet the cardinality of $\beta R^n - R^n$ is 2^c (see [4]). This brings up the question as to exactly what kind of continua may be present in the Stone-Čech remainder of R^n. In the subsequent chapters, we will provide methods for constructing different continua in $\beta R^n - R^n$. Unfortunately, the construction fails in the case $n = 1$, where the question remains largely unanswered.

An important concept in the study of the Stone-Čech compactification is that of a space being realcompact. Let $\alpha R = R \cup \{\infty\}$, the one-point compactification of the real numbers. Then any element f of $C(X)$ is a mapping of X into αR, and, therefore, has a continuous extension, βf, from βX into αR. If f is unbounded, then for some point x in $\beta X - X$, $\beta f(x) = \infty$. Define $\upsilon(X)$ to be the set of all points x in βX for which $f(x) \neq \infty$ for every $f \in C(X)$. Clearly, $X \subseteq \upsilon(X)$. We say that X is realcompact provided $X = \upsilon(X)$, i.e., for every $x \in \beta X - X$, there exists an element f, of $C(X)$, such that $\beta f(x) = \infty$. The spaces R^n, the natural numbers, and rational numbers are all examples of realcompact spaces, while any noncompact, pseudocompact space is not realcompact. In fact, every Lindelöf space is realcompact (see [4]). The importance of realcompactness becomes apparent in the theorem stating that two realcompact spaces X and Y are homeomorphic if and only if $C(X)$ and $C(Y)$ are isomorphic as rings. The proof of this theorem can be found in [4].

Using the concept of maximal ideals and the hull-kernel topology, one may obtain an alternate development of the Stone-Čech compactification which has many applications in the area of ring theory. For example, there is a one-to-one correspondence between the points of βX and the maximal ideals of the ring $C^*(X)$. The maximal ideals of $C(X)$ can also be classified using the points of βX. It is also possible to define the Stone-Čech compactification in terms of ultrafilters, which better indicates the complexity of the space βX. This development has many applications in the areas of logic and set theory.

Because of the complexity of βX, the methods of study offered by homology and homotopy prove inadequate in examining the basic topological properties of βX. Shape theory has provided new tools that have proven very useful in the study of βX, as is illustrated by the constructions in the subsequent theorems.

Shape Theory

Let $\mathcal{H}$ denote the homotopy category: the objects of $\mathcal{H}$ are topological spaces, and the morphisms are homotopy classes of continuous functions. The set of all homotopy classes of maps from X into Y will be denoted by $[X,Y]$. Let $\mathcal{W}$ represent the full subcategory of $\mathcal{H}$ whose objects are all topological spaces having the homotopy type of a CW complex. Following the development of S. Mardešić [12], we define the shape category, $\mathcal{S}$, as follows. The objects of $\mathcal{S}$ are topological spaces, and the morphisms are called shapings. If X and Y are topological spaces, a shaping $f\colon X \to Y$ is a function which assigns to every homotopy class of maps $\alpha \in [Y,K]$, where $K \in ob(\mathcal{W})$, a homotopy class $f(\alpha) \in [X,K]$ such that if $K' \in ob(\mathcal{W})$, $\alpha' \in [Y,K']$, and $\gamma \in [K,K']$, then the equality $\gamma\alpha = \alpha'$ implies $\gamma f(\alpha) = f(\alpha')$, i.e., if the diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\alpha"'] \arrow[d, "\alpha'"] \\
K \arrow[r, "\gamma"'] & K'
\end{tikzcd}$$

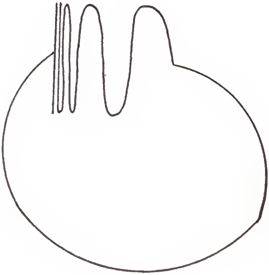
commutes, then the diagram

$$\begin{tikzcd}
 & X \arrow[dl, "f(\alpha)"'] \arrow[d, "f(\alpha')"] \\
K \arrow[r, "\gamma"'] & K'
\end{tikzcd}$$

also commutes.

If f and g are shapings, $f\colon X \to Y$, $g\colon Y \to Z$, then the *composite shaping*, $gf\colon X \to Z$ is defined by $(gf)(\alpha) = g(f(\alpha))$, where $\alpha \in [Z,K]$, and $K \in ob(\mathcal{W})$. The *identity shaping*, $1_X\colon X \to X$ is defined by $1_X(\alpha) = \alpha$. Two spaces X and Y are said to have the *same shape* ($Sh(X) = Sh(Y)$) provided there exist shapings $f\colon X \to Y$ and $g\colon Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$. In this case, f and g are called *shape equivalences*.

In [12], S. Mardešić proved that $\mathcal{S}$ is a category. We define the shape functor $S\colon \mathcal{H} \to \mathcal{S}$ as follows: $S(X) = X$ for every space X; and if $\gamma \in [X,Y]$, then $S(\gamma)\colon X \to Y$ is defined by $S(\gamma)(\alpha) = \alpha\gamma$, where $\alpha \in [Y,K]$, and $K \in ob(\mathcal{W})$. Note that not every shape morphism is induced by a continuous function. For example, let X be the graph of the sin(1/x) curve closed by an arc (the "Warsaw circle") as described:



Let Y be a circle. Then Sh(X) = Sh(Y), yet there is no continuous map from Y to X that induces this shape equivalence.

Just as covering dimension is a topological invariant under homeomorphism, we can define the notion of shape dimension which is an invariant under shape equivalence. For a compact space X, the shape dimension of X, Sd(X), is defined to be the least integer n such that there is a space Y with dim Y = n and with Y shape dominating X, i.e., there exist shapings $f: X \to Y$ and $g: Y \to X$ such that $gf = 1_X$. In [9], J. Keesling showed that for any compact set contained in the Stone-Čech remainder of a

Lindelöf space, shape dimension and covering dimension coincide (also see [18]).

Many alternate approaches to the construction of the shape category have been given. In [14], S. Mardešić and J. Segal presented a description of the shape category for compact Hausdorff spaces based on inverse systems of ANR's. While not as general as the above construction, this approach has had many applications. For example, S. Mardešić and J. Segal classified all P-adic solenoids and all n-sphere-like continua as to their shape and proved that the shape classification of 0-dimensional compacta agrees with their topological classification. In the case of compact metric spaces, K. Borsuk introduced a shape theory based on fundamental sequences, and T. Chapman formulated a description using complements of compact Z-sets in the Hilbert cube. From a categorical perspective, shape theory is closely related to the Kan and Čech extensions of a functor and to Artin and Mazur's development of Etale Homotopy.

Singular homology, singular cohomology, and the homotopy groups are invariants in the homotopy category, but, as is easily seen in the case of the "Warsaw circle" and S^1, fail to be invariants in the shape category. This problem has been remedied by the introduction of Čech homology, Čech cohomology, and the Čech homotopy groups. Here we will utilize the notion of Čech cohomology groups, as they are relatively easy to compute in our case, and two spaces with the same shape have identical Čech cohomology groups. Čech homology and the Čech homotopy groups can be defined analogously and are also shape invariants.

Čech Cohomology

In order to define the Čech cohomology groups of a topological space, we must first define the concept of direct limits.

Let D be a directed set and $\{A_\alpha\}$ a set of objects in some category. For each pair $\alpha, \beta \in D$ with $\alpha \leq \beta$, assume that there exists a morphism $\psi_{\alpha\beta}: A_\alpha \to A_\beta$ such that $\psi_{\beta\gamma} \circ \psi_{\alpha\beta} = \psi_{\alpha\gamma}$ for $\alpha \leq \beta \leq \gamma$. Then the data $\{A_\alpha, \psi_{\alpha\beta}\}_{\alpha,\beta\in D}$ is called a direct limit system, and the maps $\psi_{\alpha\beta}$ are called the connecting maps. An object $\varinjlim A_\alpha$ associated with a direct limit system is called a direct limit of $\{A_\alpha, \psi_{\alpha\beta}\}$ if there exist maps $h_\alpha: A_\alpha \to \varinjlim A_\alpha$ with $h_\alpha = h_\beta \psi_{\alpha\beta}$ such that the following holds: if there exist maps $\theta_\alpha: A_\alpha \to X$ for some object X such that $\theta_\alpha = \theta_\beta \psi_{\alpha\beta}$, then there exists a unique map $h: \varinjlim A_\alpha \to X$ such that $\theta_\alpha = hh_\alpha$. The situation is described by the following diagram:

$$\begin{tikzcd}
A_\alpha \arrow[rr, "\psi_{\alpha\beta}"] \arrow[dr] \arrow[ddr, bend right, "\theta_\alpha"'] & & A_\beta \arrow[dl] \arrow[ddl, bend left, "\theta_\beta"] \\
& \varinjlim A_\alpha \arrow[d, dashed, "h"] & \\
& X &
\end{tikzcd}$$

In the group category, it can be shown that $\varinjlim A_\alpha$ is isomorphic to $(\Sigma A_\alpha)/S$, where S is the subgroup generated by $\psi_{\alpha\beta}(a_\alpha) - a_\alpha$ (see [17]). Note also that $\varinjlim A_\alpha$ is unique up to isomorphism.

Let $\mathcal{U}(X)$ denote the collection of all numerable open coverings of a space X; $\mathcal{U}(X)$ is a directed set under the partial ordering of refinement. For each $U \in \mathcal{U}(X)$, let N(U) denote the nerve of the covering U, and $H^p(N(U),G)$ the p^{th} singular cohomology group of N(U) over an abelian group G. If V refines U, let $\pi_{UV}: N(V) \to N(U)$ be the induced projection (π_{UV} is unique up to homotopy). Then π_{UV} induces a homomorphism $\pi^*_{UV}: H^p(N(U),G) \to H^p(N(V),G)$. It is easily seen that $\{H^p(N(U),G),\pi^*_{UV}\}$ is a direct limit system over $\mathcal{U}(X)$. Now define the p^{th} Čech cohomology group of X with coefficients in G, $\check{H}^p(X,G)$, to be the direct limit group of the system $\{H^p(N(U),G),\pi_{UV}\}$.

An important property of Čech cohomology is the following: if A_n is a decreasing sequence of compact Hausdorff spaces and $A = \bigcup_{n\geq 1} A_n$, then $\check{H}^p(A,G)$ is isomorphic to $\varinjlim \check{H}^p(A_n,G)$, where the connecting maps are induced by inclusion. This is referred to as the "continuity of Čech cohomology."

We let $H^p(X)$ denote the group $\check{H}^p(X,Z)$, where Z is the additive group of integers. It is well known that $H^1(X)$ is isomorphic to $[X,S^1]$, all homotopy classes of maps from X into S^1 with the group structure induced by the group structure (complex multiplication) on S^1. This follows from the fact that S^1 is a $K(Z,1)$. (See [16].) Note that if P is a compact polyhedron, then while there is a one-to-one correspondence between the continuous maps of βX into P and those of X into P (induced by restriction), the correspondence between $[\beta X,P]$ and $[X,P]$ is not, in general, one-to-one. For example, $[\beta R,S^1]$ is uncountable, yet $[R,S^1]$ consists of just one homotopy class. In the case that $\pi_1(P)$ is finite and X is finite dimensional and normal, then A. Calder and J. Siegel have shown in [1] that $[\beta X,P]$ and $[X,P]$ are in one-to-one correspondence. This implies that for X finite dimensional and normal, $H^p(\beta X)$ is isomorphic to $H^p(X)$ for every $p > 1$. In addition, however, they have shown that for X normal and connected, $\ker\{\beta\colon [\beta X,S^1] \to [X,S^1]\}$ is isomorphic to the group $C(X)/C^*(X)$, where $C(X)$ is the additive group of real continuous functions, and $C^*(X)$ is the subgroup of bounded functions.

The importance of Čech cohomology in the study of the Stone-Čech compactification becomes apparent in the proof of the following result by J. Keesling. If X is Lindelöf and K is a nondegenerate continuum in $\beta X - X$, then K cannot have the shape of a metric continuum. The method of proof he used was to show that $H^1(K)$ contained a subgroup isomorphic to the real numbers. While this appears to severely restrict the types of continua occurring in $\beta X - X$, we will see that in the case $X = R^n$, $n \geq 2$, $\beta R^n - R^n$ contains 2^c nonhomeomorphic continua. For additional information on Čech cohomology, see [2], [5], [19].

Preliminary Theorems

Throughout the remainder of this text, we will utilize the following results:

Theorem 2.1 (Lemma 1.7 of [1]): For X normal and connected, there is an exact sequence

$$0 \to C(X)/C^*(X) \to [\beta X, S^1] \to [X, S^1] \to 0$$

where C(X) is the additive group of real valued continuous functions on X, and C*(X) is the subgroup of bounded real continuous functions.

Theorem 2.2 (Theorem 1.6 of [9]): Let $n \geq 1$ be an integer. Let X be a locally compact, σ-compact space such that for every compact set $K \subseteq X$ there is a compact set $L \subseteq X - K$ such that $\dim L \geq n$. Then the shape dimension of βX, $Sd(\beta X) \geq n$ and $Sd(\beta X - X) \geq n$.

Theorem 2.3 (Corollary 1.9 of [9]): Let X be a Lindelöf space and let K be a compact set contained in $\beta X - X$. Then $\dim(K) = Sd(K)$.

Theorem 2.4 (Theorem 1.12 of [8]): Suppose that X is realcompact and that K is a continuum contained in $\beta X - X$. Then if $f(K) = Y$ is any continuous map which is a shape equivalence, f is a homeomorphism.

CHAPTER THREE
PRELIMINARY LEMMAS

We now prove the following lemmas which will be used throughout the remainder of this paper. The proofs of these lemmas utilize the distinctive properties of the Stone-Čech compactification. Lemmas 3.1 and 3.2 are well-known but are included for the sake of completeness.

Lemma 3.1: If X is a normal space and A is a closed subset of X, then $cl_{\beta X}(A)$ is homeomorphic to βA.

Proof of Lemma 3.1

Since X is normal and A is closed in X, A is C*-embedded in X. Hence, by the construction of the Stone-Čech compactification in Chapter Two, $cl_{\beta X}(A)$ is homeomorphic to βA.

Lemma 3.2: A space X is connected if and only if βX is connected.

Proof of Lemma 3.2

Suppose X is not connected. Then $X = A \cup B$, where A and B are nonempty disjoint closed subsets of X. Define $f: X \to \{0,1\}$ by $f(A) = 0$ and $f(B) = 1$. Since A and B are disjoint and closed, f is continuous and hence has a continuous extension, βf, to all of βX. Letting $A' = (\beta f)^{-1}(\{0\})$ and $B' = (\beta f)^{-1}(\{1\})$, we obtain $\beta X = A' \cup B'$, and A' and B' are nonempty disjoint closed subsets of βX.

Conversely, if βX is not connected, there exist disjoint closed subsets A' and B' of X such that $\beta X = A' \cup B'$. Letting $A = A' \cap X$, $B = B' \cap X$, we obtain $X = A \cup B$, where A and B are disjoint closed subsets of X. Since X is dense in βX, A and B are nonempty.

Note that Lemma 3.2 is not necessarily true for an arbitrary compactification. For example, let X be the union of the two rays $(-\infty,-1]$ and $[1,\infty)$, and let αX be the one-point compactification of X. Then αX is connected but X is not connected.

We now prove the following technical lemma which will enable us to embed a collection of sets into the Stone-Čech remainder of a space, such that each set is an isolated component of the closure of the collection.

Lemma 3.3: Let $\{A_\alpha\}$ be a collection of closed subsets of a normal space X with the following property: for every α, there exists a neighborhood, N_α, of A_α, such that for any $\gamma \neq \alpha$, $N_\alpha \cap A_\gamma$ is contained in some compact subset of X. Then each $A_\alpha^* = \mathrm{cl}_{\beta X}(A_\alpha) - A_\alpha$ is an isolated component of $A = \mathrm{cl}_{\beta X}(\cup A_\alpha^*)$.

Proof of Lemma 3.3

Let M_α be a neighborhood of A_α such that $\mathrm{cl}_X(M_\alpha) \subseteq N_\alpha$. Define $f\colon \mathrm{cl}_X(M_\alpha) \cup (X - N_\alpha) \to [0,1]$ by $f(\mathrm{cl}_X(M_\alpha)) = 0$, and $f(X - N_\alpha) = 1$. Since X is normal, there is a continuous extension of f, say $\overline{f}$, to all of X. Then $\overline{f}$ can be continuously extended to all of βX, say by $\beta\overline{f}$. Since $A_\alpha \subseteq M_\alpha$,

$$\beta\overline{f}(A_\alpha) = f(A_\alpha) = 0.$$

This implies $\beta\overline{f}(\mathrm{cl}_{\beta X}(A_\alpha)) = 0$, and so $\beta\overline{f}(A^*_\alpha) = 0$. Now, for $\gamma \neq \alpha$, outside some compact set C_γ, $A_\gamma \subseteq X - N_\alpha$. Hence, A^*_γ is contained in the closure of $A_\gamma \cap (X - C_\gamma) \subseteq X - N_\alpha$. This implies

$$\beta\overline{f}(A^*_\gamma) = \beta\overline{f}(X - N_\alpha) = f(X - N_\alpha) = 1.$$

Therefore, $\beta\overline{f}(\bigcup_{\gamma\neq\alpha} A^*_\gamma) = 1$ and hence, $\beta\overline{f}(\mathrm{cl}_{\beta X}(\bigcup_{\gamma\neq\alpha} A^*_\gamma)) = 1$. By normality, there exist open sets U and V in βX with $U \cap V = \emptyset$, $A^*_\alpha \subseteq U$, and $\mathrm{cl}_{\beta X}(\bigcup_{\gamma\neq\alpha} A^*_\gamma) \subseteq V$. Hence, A^*_α is an isolated component of

$$A = (\mathrm{cl}_{\beta X}(\bigcup_{\gamma\neq\alpha} A^*_\gamma)) \cup A^*_\alpha.$$

Next, we prove a surprising result concerning the dimension of neighborhoods of points in the Stone-Čech remainder of R^m.

Lemma 3.4: Suppose $\{B_k\}_{k=1}^{\infty}$ is a decreasing sequence of closed, locally n-dimensional sets contained in R^m.

Then for any point x contained in $B = \bigcap_{k\geq 1} B_k^*$, any neighborhood of x in B has dimension n.

Proof of Lemma 3.4

Let V be a neighborhood of x in B. Then $V = W \cap B$, where W is open in βR^m. Since $x \in W$, $W \cap R^m$ is an open subset of R^m that intersects every B_k in an unbounded set. Consider $W \cap B_k$, and let D_k be a closed n-ball contained in $W \cap B_k$ such that D_k lies outside the m-1 dimensional sphere S_k centered at the origin with radius k. This is possible because for any compact set C in R^m, $W \cap B_k \cap (R^m - C)$ must contain an n-dimensional subset with interior points, since B_k is locally n-dimensional. Let $D = \bigcup_{k\geq 1} D_k$. Then D is a closed subset of R^m. Hence, $D^* = \beta D - D$ is contained in $\beta R^m - R^m$, by Lemma 3.1.

We will show that D^* is contained in V. Clearly, $D^* \subseteq W$. By the construction of D, and the fact that $B_j \supseteq B_{j+1}$ for every integer j, we have that for $k \geq 1$, every point of D not in S_k is contained in B_k. Hence, $D^* \subseteq B_k^*$ for every $k \geq 1$, which implies $D^* \subseteq \bigcap_{k\geq 1} B_k^* = B$. Hence, $D^* \subseteq B \cap W = V$. Now, $Sd(D^*) = Sd(\beta D - D) \geq n$ by Theorem 2.2. By Theorem 2.3, $\dim(D^*) \geq n$. Since $\dim(D^*)$ is also $\leq n$, it follows that $\dim(D^*) = n$. Hence, V has dimension n.

Note that as a corollary of Lemma 3.4 we have that $\beta R^n - R^n$ is locally n-dimensional. (Let $B_k = R^n$ for every integer k.)

The following lemma provides an upper bound for the number of closed subsets that a separable space may contain. This will then give an upper bound for the number of closed subsets of βR^n.

Lemma 3.5: If X is a (completely regular) space with a countable dense subset, then X has at most 2^c closed subsets. In particular, if X has a countable dense subset, then βX contains at most 2^c closed subsets.

Proof of Lemma 3.5

We begin by showing that X has a basis of at most c open sets. Let A be a countable dense subset of X. Consider the set of all continuous functions from X into the closed interval [0,1]. Call this set $\mathcal{F}$. Each function in $\mathcal{F}$ is completely determined by its action on the elements of A. Hence, there are $(\text{card}([0,1]))^{(\text{card}(A))} = c^{\aleph_0} = c$ such functions. Now, consider the collection $\mathcal{U}$, of all open sets of X of the form $f^{-1}(W)$, where $f \in \mathcal{F}$, and W is an open subset of [0,1]. We claim that $\mathcal{U}$ forms a basis for the topology of X.

Let V be an open subset of X and $x \in V$. By complete regularity, there is a continuous function $f: X \to [0,1]$ such that $f(x) = 0$, and $f(X - V) = 1$. Then, $f^{-1}([0,1/2))$

is an open subset of V containing the point x, and $f^{-1}([0,1/2))$ is an element of $\mathcal{U}$. Hence, $\mathcal{U}$ forms a basis for the topology of X, and $\mathcal{U}$ has cardinality $c^2 = c$.

Now, every open subset of X is the union of basic open sets of $\mathcal{U}$. Hence, there are at most 2^c open subsets of X. Since every closed set contained in X is the complement of an open set, this says that there are at most 2^c closed subsets of X.

To prove the second part, one needs only to note that if A is a countable dense subset of X, then A is also dense in βX, as $cl_{\beta X}(A) = cl_{\beta X}(cl_X(A)) = cl_{\beta X}(X) = \beta X$. Hence, βX contains at most 2^c closed subsets.

Note that in the above lemma, if X is not compact, then βX consists of 2^c points. This follows from the fact that a copy of the natural numbers, N, can be C*-embedded in X, and βN has cardinality 2^c (see [20]).

CHAPTER FOUR
CONTINUA IN THE STONE-ČECH REMAINDER OF R^n, $n \geq 3$

In this chapter, we shall prove three theorems concerning the types of continua and compacta that appear in the Stone-Čech remainder of R^n for $n \geq 3$. The construction is carried out in dimension 3; these results imply the results for higher dimensions. We begin by constructing c continua in the Stone-Čech remainder of R^3 with the property that any two have nonisomorphic first Čech cohomology groups.

Theorem 4.1: There are c subcontinua of $\beta R^3 - R^3$ which have nonisomorphic first Čech cohomology groups.

Proof of Theorem 4.1

Consider the collection $\{P_a : a \in A\}$, where each P_a is a sequence of prime numbers with the following properties: there are an infinite number of distinct primes in P_a, and each prime occurs an infinite number of times; if $a, b \in A$ with $a \neq b$, then there is a prime occurring in P_a which is not in P_b, or a prime in P_b which is not in P_a; and card $A = c$. Let Σ_a be the solenoid corresponding to the sequence P_a, and let $B_a = H^1(\Sigma_a)$. We know that B_a is isomorphic to $\{\frac{m}{p_1 p_2 \cdots p_k} : m \in Z,\ p_i \in P_a\}$. (For properties of solenoids, see [7].)

The solenoid Σ_a may be described as follows: let $P_a = \{p_1, p_2, p_3, \ldots\}$; Σ_a is the intersection of a decreasing tower of solid tori $\{T_n\}$ in R^3 with the properties that

(i) $T_{n+1} \subseteq T_n$ for every $n \in Z^+$;

(ii) $\lim_{n\to\infty}$ [length of cross section of T_n] $= 0$; and

(iii) T_{n+1} is wrapped p_n times around the hole of T_n.

Also, let $p, q \in T_1$ so that the distance from p to q is maximal, and specify that T_n passes through p and q for every n. (See Figure 1.)

Position Σ_a in R^3 so that $p = (0,0,0)$ and $q = (0,0,1)$. Define $f: R^3 \to R^3$ by $f(x,y,z) = (x,y,z+1)$, and let $A = \bigcup_{n\geq 0} f^n(\Sigma_a)$. Hence, A is the union of a countable number of copies of Σ_a placed end to end. (See Figure 2.) Now $H^1(A) = \prod_{n\geq 0} H^1(f^n(\Sigma_a)) = \Pi H^1(\Sigma_a)$ (the countable infinite product of copies of $H^1(\Sigma_a)$), and so we have $H^1(A) = \Pi B_a$.

Let $A_n = \bigcup_{i\geq n} f^i(\Sigma_a)$, i.e., A_n is the closure of A with the first n copies of Σ_a deleted. Since A and A_n are closed subsets of R^3, βA and βA_n are contained in βR^3, by Lemma 3.1. Also, A_n is connected implies that βA_n is connected, by Lemma 3.2. Hence, $\beta A - A = \bigcap_{n\geq 0} \beta A_n$ is a continuum in $\beta R^3 - R^3$. Let $A^* = \beta A - A$. We now wish to compute $H^1(A^*)$.

By Theorem 4.1, there is an exact sequence

$$0 \to C(X)/C^*(X) \to [\beta X, S^1] \to [X, S^1] \to 0,$$

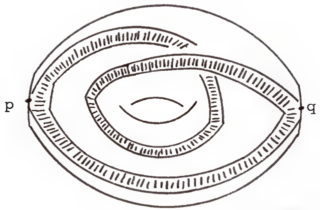


Figure 1

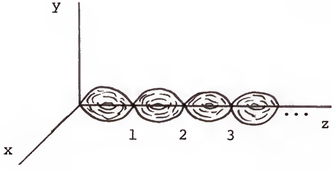


Figure 2

where $C(X)$ is the additive group of real continuous functions on X, and $C^*(X)$ is the subgroup of bounded functions. Since $A^* = \bigcap_{n\geq 0} \beta A_n$, by the continuity of Čech cohomology, $H^1(A^*) = \varinjlim H^1(\beta A_n)$, where the bonding maps are induced by inclusion, $i_n^*\colon H^1(\beta A_n) \to H^1(\beta A_{n+1})$. For each n, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C(A_n)/C^*(A_n) \arrow[r] \arrow[d, "i_n^*"] & {[\beta A_n, S^1]} \arrow[r] \arrow[d, "i_n^*"] & {[A_n, S^1]} \arrow[r] \arrow[d, "i_n^*"] & 0 \\
0 \arrow[r] & C(A_{n+1})/C^*(A_{n+1}) \arrow[r] & {[\beta A_{n+1}, S^1]} \arrow[r] & {[A_{n+1}, S^1]} \arrow[r] & 0
\end{tikzcd}$$

This diagram gives rise to the following exact sequence:

$$0 \to \varinjlim C(A_n)/C^*(A_n) \to \varinjlim [\beta A_n, S^1] \to \varinjlim [A_n, S^1] \to 0.$$

Since $[X, S^1] \simeq H^1(X)$, we have $\varinjlim [\beta A_n, S^1] \simeq \varinjlim H^1(\beta A_n) \simeq H^1(A^*)$, and $\varinjlim [A_n, S^1] \simeq \varinjlim H^1(A_n)$, where the bonding maps are i_n^*. Hence, we have the following exact sequence:

$$0 \to \varinjlim C(A_n)/C^*(A_n) \to H^1(A^*) \to \varinjlim H^1(A_n) \to 0$$

We will now evaluate these direct limits.

Since A_n differs from A_{n+1} by a set of compact closure, $i_n^*\colon C(A_n)/C^*(A_n) \to C(A_{n+1})/C^*(A_{n+1})$ is an isomorphism. Hence, an easy computation shows that $\varinjlim C(A_n)/C^*(A_n)$ is isomorphic to $C(A_1)/C^*(A_1)$. Since $C(A_1)/C^*(A_1)$ is a torsion free divisible group, $C(A_1)/C^*(A_1)$ is isomorphic to a direct sum of copies of Q, the rational numbers (see [17]). Therefore,

$$\varinjlim C(A_n)/C^*(A_n) \simeq \bigoplus_{c} Q.$$

Now consider $\varinjlim H^1(A_n)$. As before, $H^1(A_n)$ is isomorphic to ΠB_a, the countable infinite product of copies of B_a. The bonding map $i_n^*: H^1(A_n) \to H^1(A_{n+1})$ is defined by $i_n^*((x_1,x_2,x_3,\ldots)) = (x_2,x_3,\ldots)$, where $x_i \in B_a$. Now $\varinjlim H^1(A_n)$ is isomorphic to $(\Sigma H^1(A_n))/S = (\Sigma(\Pi B_a))/S$, where S is the subgroup generated by $i_n^*(y_n) - y_n$, $y_n \in H^1(A_n)$. (See [17], page 29.) Define a map $g: \Pi B_a \to (\Sigma(\Pi B_a))/S$ by $g(a) = (a,0,0,\ldots) + S$. We now show that g is an onto homomorphism with kernel ΣB_a. Clearly, g is a homomorphism. To show that g is onto, let $(a_1,a_2,\ldots,a_n,0,0,\ldots) \in \Sigma(\Pi B_a)$, where $a_i \in \Pi B_a$, say $a_i = (a_i^1,a_i^2,a_i^3,\ldots)$. We have that

$$\begin{aligned}(a_1,a_2,\ldots,a_n,0,0,\ldots) &= (a_1,0,0,\ldots) + (0,a_2,0,0,\ldots)\\ &\quad + \ldots + (0,0,\ldots,0,a_n,0,\ldots).\end{aligned}$$

Now, two elements in $\Sigma(\Pi B_a)$ are equivalent in $\Sigma(\Pi B_a))/S$ if and only if their difference is in S. Hence,

$$\begin{aligned}(0,(a_2^1,a_2^2,\ldots),0,0,\ldots) &\equiv ((0,a_2^1,a_2^2,\ldots),0,0,\ldots) \text{ and}\\ (0,0,(a_3^1,a_3^2,\ldots),0,\ldots) &\equiv (0,(0,a_3^1,a_3^2,\ldots),0,0,\ldots)\\ &\equiv ((0,0,a_3^1,a_3^2,\ldots),0,0,\ldots).\end{aligned}$$

In general,

$$(0,\ldots,0,(a_i^1,a_i^2,\ldots),0,\ldots) \equiv ((\underbrace{0,\ldots,0}_{i-1},a_i^1,a_i^2,\ldots),0,0,\ldots).$$

Therefore, $(a_1,a_2,\ldots,a_n,0,0,\ldots) \equiv (a_1',0,0,\ldots)$ where $a_1' \in \Pi B_a$, and $g(a_1') = (a_1,a_2,\ldots,a_n,0,0,\ldots)$. Hence, g is onto. To show that the kernel of g is ΣB_a, it suffices to show that if $a = (a_1, a_2, a_3, \ldots)$ is an element of ΠB_a, then $(a, 0, 0, \ldots)$ is equivalent to 0 in $(\Sigma(\Pi B_a))/S$ if and only if a has only a finite number of nonzero entries. If a has a finite number of nonzero entries, say $a = (a_1, a_2, \ldots, a_n, 0, 0, \ldots)$, then

$$\begin{aligned}(a,0,0,\ldots) &\equiv (0,(a_2,a_3,\ldots,a_n,0,\ldots),0,0,\ldots)\\ &\equiv (0,0,(a_3,a_4,\ldots,a_n,0,\ldots),0,0,\ldots)\\ &\;\;\vdots\\ &\equiv (0,0,\ldots,0,(a_n,0,\ldots),0,\ldots)\\ &\equiv (0,0,\ldots,0,(0,0,\ldots),0,\ldots)\\ &= 0.\end{aligned}$$

Conversely, if a has an infinite number of nonzero entries, we see that $(a, 0, 0, \ldots)$ is never equivalent to $(0, 0, \ldots)$. Therefore, the kernel of g is ΣB_a.

Now, since g is an onto homomorphism with kernel ΣB_a, g induces an isomorphism $(\Sigma(\Pi B_a))/S \simeq (\Pi B_a)/(\Sigma B_a)$. Letting $\tilde{\Pi} B_a$ denote the group $(\Pi B_a)/(\Sigma B_a)$, we obtain the result that $\varinjlim H^1(A_n) \simeq \tilde{\Pi} B_a$.

By these two evaluations, we get the following exact sequence:

$$0 \to \bigoplus_c Q \to H^1(A^*) \to \tilde{\Pi} B_a \to 0.$$

Since $\underset{c}{\oplus} Q$ is divisible, the sequence splits (see [6]), and $H^1(A^*) \simeq \tilde{\Pi}B_a \oplus (\underset{c}{\oplus} Q)$. Thus we have constructed a continuum A^* in $\beta R^3 - R^3$ with $H^1(A^*) \simeq \tilde{\Pi}B_a \oplus (\underset{c}{\oplus} Q)$.

For any $b \in A$, $b \neq a$, we can construct a continuum B^* in $\beta R^3 - R^3$ in an analogous manner such that $H^1(B^*)$ is isomorphic to $\tilde{\Pi}B_b \oplus (\underset{c}{\oplus} Q)$. We will show that for $a,b \in A$ such that $a \neq b$, $H^1(A^*)$ is not isomorphic to $H^1(B^*)$.

Suppose $h\colon H^1(A^*) \to H^1(B^*)$ is an onto isomorphism. Then $\tilde{\Pi}B_a \oplus (\oplus Q) \simeq \tilde{\Pi}B_b \oplus (\oplus Q)$. Either $P_a - P_b \neq \emptyset$, or $P_b - P_a \neq \emptyset$, so without loss of generality assume that $P_a - P_b \neq \emptyset$, and let $p \in P_a - P_b$. Consider the class represented by the element $(1,1,1,\ldots) \oplus (0)$ of $\tilde{\Pi}B_b \oplus (\oplus Q)$. There is an element $(a_1,a_2,a_3,\ldots) \oplus (q)$, where $(a_1,a_2,\ldots) \in \Pi B_a$ and $(q) \in \oplus Q$ such that

$$h((a_1,a_2,\ldots) \oplus (q)) \equiv (1,1,1,\ldots) \oplus (0).$$

Now, a_i/p is an element of B_a for every i, so $\frac{1}{p}((a_1,a_2,\ldots) \oplus (q))$ is an element of $\Pi B_a \oplus (\oplus Q)$. Let $(c_1,c_2,\ldots) \oplus (q')$ be a representative for the equivalence class of $h(\frac{1}{p}((a_1,a_2,\ldots) \oplus (q)))$. Then,

$$\begin{aligned} p((c_1,c_2,\ldots) \oplus (q')) &\equiv ph(\tfrac{1}{p}((a_1,a_2,\ldots) \oplus (q))) \\ &\equiv h(\tfrac{p}{p}((a_1,a_2,\ldots) \oplus (q))) \\ &\equiv h((a_1,a_2,\ldots) \oplus (q)) \\ &\equiv (1,1,1,\ldots) \oplus (0). \end{aligned}$$

This implies that $p(c_1,c_2,\ldots)$ is equivalent to $(1,1,\ldots)$, which implies $p(c_1,c_2,\ldots) - (1,1,1,\ldots)$ is an element of ΣB_a.

Hence, $p(c_1,c_2,\ldots) - (1,1,1,\ldots) = (b_1,b_2,\ldots,b_n,0,0,\ldots)$, with equality in ΠB_b. This implies

$$p(c_1,c_2,\ldots) = (b_1+1,b_2+1,\ldots,b_n+1,1,1,1,\ldots).$$

So, for $m \geq n + 1$, $pc_m = 1$, which implies $c_m = 1/p$. But p does not occur in P_b, so $1/p$ is not an element of B_b, and, hence, c_m cannot be an element of B_b, which is a contradiction. Hence, $H^1(A^*)$ is not isomorphic to $H^1(B^*)$.

Since there are c choices for $a \in A$, we have constructed c continua in $\beta R^3 - R^3$ with nonisomorphic first Čech cohomology groups. Since two spaces with nonisomorphic Čech cohomology groups have different shapes, we have the following corollary.

Corollary 4.1: There are c continua in $\beta R^3 - R^3$, no two of which have the same shape.

In the next theorem, we will modify the construction in Theorem 4.1 to form 2^c compacta in $\beta R^3 - R^3$ with different shapes. The proof will depend heavily on Lemma 3.3, and the fact that a shape equivalence between two spaces induces a shape equivalence between the isolated components of the spaces.

Theorem 4.2: There are 2^c compacta in $\beta R^3 - R^3$, no two of which have the same shape.

Proof of Theorem 4.2

Theorem 4.2 is a continuation of Theorem 4.1. Suppose $\mathcal{A}$, A, and A* are as in the proof of Theorem 4.1. For each $a \in \mathcal{A}$, we have constructed a continuum A* in $\beta R^3 - R^3$ such that for $a \neq b$, $Sh(A^*) \neq Sh(B^*)$. Now for each subset of $\mathcal{A}$ of cardinality c, we will construct a compactum in $\beta R^3 - R^3$ such that if $S_1, S_2 \subseteq \mathcal{A}$, $S_1 \neq S_2$, and card S_1 = card $S_2 = c$, then the corresponding compacta will have different shapes. Since there are 2^c subsets of $\mathcal{A}$ of cardinality c, this will exhibit 2^c nonshape equivalent compacta in $\beta R^3 - R^3$.

Let $S \subseteq \mathcal{A}$ such that card $S = c$. There is a one-to-one correspondence between elements of S and real numbers r such that $0 \leq r < 2\pi$. So, to each element a of S, we correspond to a unique number $r_a \in [0,2\pi)$. Let $h_{r_a} : R^3 \to R^3$ be a rotation of the y-z plane r_a radians. Define $A_r = h_{r_a}(A)$, where A is as defined above. As before,

$$H^1(A_r^*) = \tilde{\Pi} B_a \oplus (\bigoplus_c Q),$$

where $A_r^* = \beta A_r - A_r$. Let $C_S = \overline{\bigcup_{a \in S} A_r^*}$ (closure in βR^3). Then C_S is a compact subset of $\beta R^3 - R^3$.

Now, choose $a \in S$ and consider A_r^*. Let N_a be a neighborhood of the ray $h_{r_a}(\{(0,0,z) : z \in R^+\})$ of radius 2. By construction, $A_r \subseteq N_a$. For any $b \in S$, $b \neq a$, there is a (bounded) neighborhood about the origin, such that $N_a \cap B_r$ is contained in this neighborhood. Hence, by Lemma 3.3, A_r^* is an isolated component of C_S.

Note that these are the only isolated components, for if $X \subseteq C_S - \bigcup_{a \in S} A_r^*$, then any open set containing X also contains points of $\bigcup_{a \in S} A_r^*$, since every point of X is a limit point of $\bigcup_{a \in S} A_r^*$.

Now, for $S_1, S_2 \subseteq A$ with $S_1 \neq S_2$ and card S_1 = card $S_2 = c$, the shape of C_{S_1} is different from the shape of C_{S_2}. This follows from the fact that if $Sh(C_{S_1}) = Sh(C_{S_2})$, then each isolated component of C_{S_1} is shape equivalent to an isolated component of C_{S_2}. Either $S_1 - S_2 \neq \emptyset$, or $S_2 - S_1 \neq \emptyset$, so without loss of generality assume that $S_1 - S_2 \neq \emptyset$, and let $a \in S_1 - S_2$. Then A_r^* is an isolated component of C_{S_1} which is not shape equivalent to any isolated component of C_{S_2}, which implies that $Sh(C_{S_1}) \neq Sh(C_{S_2})$.

Hence, there are 2^c compacta in $\beta R^3 - R^3$ no two of which have the same shape. Since there are at most 2^c compacta in βR^3, there are exactly 2^c compacta in $\beta R^3 - R^3$ no two of which have the same shape.

Corollary 4.2: For $n \geq 3$, there are 2^c compacta in $\beta R^n - R^n$, no two of which have the same shape.

Proof of Corollary 4.2

Corollary 4.2 follows from Lemmas 3.1 and 3.5 and the fact that R^3 is a closed subset of R^n for $n \geq 3$.

In Theorem 4.3, we will modify the construction in Theorem 4.2 by adding a subset of $\beta R^3 - R^3$ to each of the above compacta already formed. This will transform the compacta into continua, and a dimension theoretic argument will show that any two of them are not homeomorphic.

Theorem 4.3: There are 2^c nonhomeomorphic continua in $\beta R^3 - R^3$.

Proof of Theorem 4.3

As in the proof of Theorem 4.2, let $S \subseteq A$ such that card $S = c$; $A_r = h_{r_a}(A)$; and $C_S = \overline{\bigcup_{a\in S} A_r^*}$.

Consider a plane P tangent to each solenoid of $\bigcup_{a\in S} A_r$. This is possible since all of the solenoids in $\cup A_r$ are uniform, so any plane parallel to the y-z plane and tangent to one of the solenoids will be tangent to all of them. Then P is a closed subset of R^3, so $P^* = \beta P - P \subseteq \beta R^3 - R^3$. Let $X = C_S \cup P^*$.

We will first show that X is a continuum in $\beta R^3 - R^3$. Since C_S and P^* are closed in $\beta R^3 - R^3$, $C_S \cup P^*$ is closed and, hence, compact. Now, suppose $\overline{\cup A_r^*} \cup P^* \subseteq U \cup V$, where U and V are open in βR^3 and $U \cap V = \emptyset$. Since P^* is connected, $P^* \subseteq U$ or $P^* \subseteq V$, say $P^* \subseteq U$. Since $P^* \cap A_r^* \neq \emptyset$, $A_r^* \cap U \neq \emptyset$. Then, since A_r^* is connected, $A_r^* \subseteq U$ for every A_r^*. Hence, $\cup A_r^* \subseteq U$, which implies that $(\cup A_r^*) \cup P^* \subseteq U$ and, hence, $\overline{\cup A_r^*} \cup P^* \subseteq U$. Therefore, $\overline{\cup A_r^*} \cup P^* = X$ is connected, and X is a continuum.

Now suppose $C_T = \overline{\bigcup_{b \in T} B_r^*}$ is the result of a collection of solenoids corresponding to the subset T of A, where card T = c and T ≠ S. Then $Y = C_T \cup P^*$ is also a continuum in $\beta R^3 - R^3$.

We will shown that X and Y are not homeomorphic. The method will be as follows. If h is a homeomorphism from X onto Y, then we will show that $h(C_S) = C_T$, which implies that C_S and C_T are homeomorphic, contradicting the fact that C_S and C_T have different shapes by Theorem 4.2 and, therefore, are not homeomorphic.

So, suppose h is a homeomorphism from X onto Y. To show that $h(C_S) = C_T$, first consider an element x of A_r^* such that x is not an element of P*.

As in Theorem 4.2, A_r^* is an isolated component of C_S, so there is a set V, open in βR^3 such that $A_r^* \subseteq V$ and $(C_S - A_r^*) \cap V = \emptyset$. Since X is normal, there exists an open set U containing x such that $U \cap P^* = \emptyset$. Then $U \cap V \subseteq A_r^*$ and $x \in U \cap V$. Also, $(U \cap V) \cap P^* = \emptyset$.

Suppose h(x) is not an element of C_T. Then h(x) is contained in $P^* \cap (Y - C_T)$. Hence, there is an open set N of Y such that $h(x) \in N$ and $N \cap C_T = \emptyset$. Consider the set $[h(U \cap V)] \cap N$. This is an open set of h(x) contained in P*. By Lemma 3.4, there is a set W, open in P*, such that $\overline{W} \subseteq h(U \cap V) \cap N$ and $\dim(\overline{W}) = 2$. Since P* is closed, $\overline{W}$ is a closed subset of Y that contains h(x) with $\dim(\overline{W}) = 2$ and $\overline{W} \subseteq h(U \cap V)$. Now $h^{-1}(\overline{W}) \subseteq U \cap V$ is a closed subset of A_r^* containing x. Since h^{-1} is a homeomorphism, h^{-1} preserves dimension, so $h^{-1}(\overline{W})$ has dimension 2. This says that A_r^* has a closed subspace of dimension 2, which implies $\dim(A_r^*) \geq 2$. But Theorems 2.2 and 2.3 imply that $\dim(A_r^*) = 1$, since solenoids have dimension 1. This contradicts the fact that h is a homeomorphism. Hence, $h(x) \in C_T$.

Now, suppose that x is an element of $A_r^* \cap P^*$. We will show that x is a limit point of $A_r^* \cap (X - P^*)$. Then, by the above argument, since $h(A_r^* \cap (X - P^*)) \subseteq C_T$, it follows that $h(x) \in C_T$.

Let U be an open set in $\beta R^3 - R^3$ containing x. There is a set W, open in $\beta R^3 - R^3$, such that $x \in W \subseteq \overline{W} \subseteq U$. Now, $W = (\beta R^3 - R^3) \cap V$, where V is open in βR^3. Since V is an open set containing $x \in A_r^* = \beta A_r - A_r$, $V \cap A_r \neq \emptyset$. This implies that V intersects an infinite number of solenoids of A_r. (If V intersects only a finite number, say Σ is the union of the solenoids that intersect V, then $V \cap (R^3 - \Sigma)$ is an open set of x missing A_r.)

Now, let $\{x_n\colon n \geq 1\} \subseteq A_r \cap V \cap (R^3 - P)$ such that the norm of x_n tends to infinity as n tends to infinity. This is possible since $V \cap R^3$ is open, and $A_r \cap P$ is a countable set. Let $y \in \beta(\{x_n\colon n \geq 1\}) - \{x_n\colon n \geq 1\} \subseteq \beta R^3 - R^3$. Since $x_n \in A_r$ for every n, $y \in \beta A_r - A_r$. Now, define f on $P \cup \{x_n\colon n \geq 1\}$ by $f(P) = 0$ and $f(x_n) = 1$ for every n. Since $P \cup \{x_n\colon n \geq 1\}$ is closed in R^3, there is a continuous extension of f to all of R^3, say $\overline{f}$. Then $\overline{f}$ can be extended continuously to βR^3, say by $\beta\overline{f}$. Now $\beta\overline{f}(x_n) = 1$ for every n implies that $\beta\overline{f}(y) = 1$. Since $\beta\overline{f}(P) = 0$,

$\beta\overline{f}(\overline{P}) = 0$. Hence, $y \notin \beta P$. Also, $x_n \in V$ for every n, which implies that $y \in \overline{V} - R^3$, and hence $y \in \overline{W} \subseteq U$. Therefore, $U \cap (A^*_r - P^*) \neq \emptyset$, which implies that x is a limit point of $A^*_r \cap (X - P^*)$. Hence, $h(x) \in C_T$.

By the above arguments, $h(A^*_r) \subseteq C_T$ for every A^*_r. Then $h(\cup A^*_r) \subseteq C_T$, which implies $h(\overline{\cup A^*_r}) \subseteq \overline{C}_T = C_T$, and, hence, $h(C_S) \subseteq C_T$. Similarly, $h^{-1}(C_T) \subseteq C_S$, which implies $C_T \subseteq h(C_S)$. Therefore, $h(C_S) = C_T$ and C_S and C_T are homeomorphic. This contradicts Theorem 4.2, since $Sh(C_S) \neq Sh(C_T)$. Hence, X and Y are not homeomorphic.

By Theorem 4.2, there are 2^c choices for X, and since no two of them are homeomorphic, there are 2^c nonhomeomorphic continua in $\beta R^3 - R^3$.

Corollary 4.2: For $n \geq 3$, $\beta R^n - R^n$ contains 2^c nonhomeomorphic continua.

Corollary 4.3: Let X and Y be as in the proof of Theorem 4.3. Then there does not exist a continuous map $f: X \to Y$ that induces a shape equivalence. In particular, X and Y are not homotopic.

Proof of Corollary 4.3

By Theorem 2.4, if f is a continuous map, $f: X \to Y$, which induces a shape equivalence, then f is a homeomorphism, contradicting Theorem 4.3.

Note that Corollary 4.3 does not imply that X and Y are not shape equivalent, since there are shape morphisms that are not induced by continuous functions, as we saw in the case of the Warsaw circle and S^1.

Also, note that since solenoids cannot be embedded in R^2, the same argument fails in the case $n = 2$. In fact, the method of Theorem 4.1 fails in general for R^2, since the cohomology of a continuum in the plane is either 0 or a direct sum of copies of Z, the integers.

CHAPTER FIVE
CONTINUA IN THE STONE-ČECH REMAINDER OF R^2

We will now establish the fact that $\beta R^2 - R^2$ also contains 2^c nonhomeomorphic continua. The construction in this case is much more complicated than in the case of R^3, and the proof depends heavily on Theorems 2.2 and 2.3, and Lemma 3.3. Throughout the proof, the image under a rotation of R^2 of a set in R^2 of the form

$$\{(x,y): x \geq 0,\ \alpha \leq y \leq \gamma;\ \alpha,\gamma \in R\}$$

will be called a _thickened_ _ray_.

Theorem 5.1: There are 2^c nonhomeomorphic continua in $\beta R^2 - R^2$.

Proof of Theorem 5.1

For the sake of clarity, we consider first the construction of c nonhomeomorphic continua in $\beta R^2 - R^2$. We will then apply these arguments and results in the construction of 2^c nonhomeomorphic continua in $\beta R^2 - R^2$.

Consider a collection $\{P_a : a \in A\}$ where each P_a is an infinite subset of positive integers; for $a \neq b$, either $P_a - P_b \neq \emptyset$ or $P_b - P_a \neq \emptyset$; and card $A = c$. For $p \in P_a$, consider the two rays $\{(x,y) : x \geq 0,\ y = 1/p\}$ and

$\{(x,y) : x \geq 0, y = 1/(p+1)\}$. Between these rays, consider p disjoint thickened rays, say $T_a(p,n)$, where $n = 1,2,\ldots p$, and labeled so that if $n_1 < n_2$, the y-coordinate of any point in $T_a(p,n_1)$ is greater than the y-coordinate of points in $T_a(p,n_2)$.

Let $L(n) = \{(x,y) : x \geq 0, y = 1/n\}$, and let $C(n) = \{(x,y) : x^2 + y^2 = n, x \geq 0, 0 \leq y \leq 1\}$. The situation is similar to the one described in Figure 3.

The continuum X will be formed as follows. Let T_a denote the union of the Stone-Čech remainders of the thickened rays $T_a(p,n)$, L the union of the remainders of the rays $L(n)$, and C the remainder of the union of the curves $C(n)$. Then X will be the closure in βR^2 of the union of these sets, i.e., $X = \overline{T}_a \cup \overline{L} \cup C$. (Note that X is not the Stone-Čech remainder of the closure in R^2 of the union of the rays and curves.) We first show that X is a continuum in $\beta R^2 - R^2$.

Suppose $X \subseteq U \cup V$, where U and V are open subsets of βR^2, and $U \cap V = \emptyset$. Let $p \in P_a$, $1 \leq m \leq p$, and consider the Stone-Čech remainder of $T_a(p,m)$, $T_a^*(p,m)$. Since $T_a^*(p,m)$ is connected, it is contained in U or in V, say $T_a^*(p,m) \subseteq U$. Then, since $C(n)$ intersects $T_a(p,m)$ for every $n \geq 1$, each component of the Stone-Čech remainder of $\bigcup_{n\geq 1} C(n)$ must also intersect $T_a^*(p,m)$, and hence, must be contained in U. Therefore, $C \subseteq U$. Now, for any $p' \in P_a$, $1 \leq m' \leq p'$, $T_a(p',m')$ intersects $C(n)$ for every $n \geq 1$, so $T_a^*(p',m') \cap U \neq \emptyset$. Hence, $T_a^*(p',m') \subseteq U$.

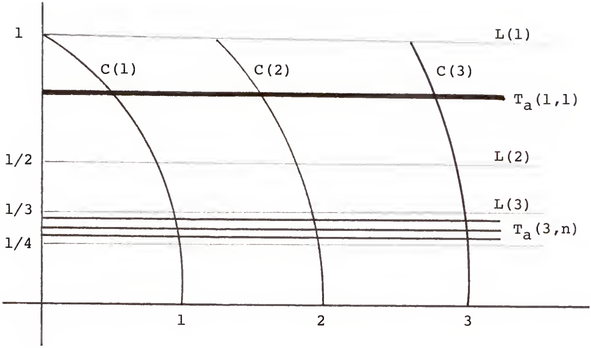


Figure 3

Therefore, $T_a \subseteq U$, which implies $\overline{T}_a \subseteq U$. Similarly, each $L(m)$ intersects $C(n)$ for every $n \geq 1$, and hence, $L \subseteq U$. This implies $\overline{L} \subseteq U$, and so,

$$X = \overline{T}_a \cup \overline{L} \cup C \subseteq U.$$

Therefore, X is connected. Since X is also compact, X is a continuum.

For a different subset P_b of positive integers, we define T_b analogously, and let $Y = \overline{T}_b \cup \overline{L} \cup C$. Then Y is also a continuum in $\beta R^2 - R^2$.

We will show that X and Y are not homeomorphic. Suppose h is a homeomorphism from X onto Y. We begin by showing that $h(\overline{T}_a) = \overline{T}_b$.

Suppose $x \in T_a^*(p,n) = \beta(T_a(p,n)) - T_a(p,n)$ for some $p \in P_a$, $1 \leq n \leq p$, so that x is not an element of $\overline{C - T_a}$. Then, since $T_a^*(p,n) \cap \overline{L} = \emptyset$, there is a neighborhood $N(x)$ of x in X such that $N(x) \subseteq T_a^*(p,n)$. Suppose $h(x)$ is not an element of $\overline{T}_b$. Then $h(x) \in \overline{L}$ or $h(x) \in C - (\overline{L} \cup \overline{T}_b)$. But $C - (\overline{L} \cup \overline{T}_b)$ is open in Y, so each point of $C - (\overline{L} \cup \overline{T}_b)$ has a neighborhood of dimension ≤ 1, since $\dim(C) = 1$. Since any neighborhood of x has dimension 2, by Lemma 3.4, $h(x)$ cannot be an element of $C - (\overline{L} \cup \overline{T}_b)$. Hence, $h(x) \in \overline{L}$. Then $h(N(x))$ is a neighborhood of $h(x)$, which implies there is a point $y \in L$ such that $y \in h(N(x))$. But since $y \in L$, y has neighborhoods of dimension ≤ 1, while every neighborhood of $h^{-1}(y)$ has dimension 2, since $h^{1}(y) \in N(x)$ and $N(x) \subseteq T_a^*(p,n)$. This is a contradiction, and so it follows that $h(x) \in \overline{T}_b$.

Next, we will show that every point of $T_a^*(p,n)$ is a limit point of such points x. From this it will follow that $h(T_a^*(p,n)) \subseteq \overline{T}_b$.

Let $x \in T_a^*(p,n)$, and let U be an open subset of X containing x. There is a set W, open in X, such that $x \in W \subseteq \overline{W} \subseteq U$. Then $W = X \cap V$, where V is open in βR^2. Now, $V \cap R^2$ is an open subset of R^2 that intersects $T_a(p,n)$ in an unbounded set. Let $\{x_n : n \geq 1\} \subseteq V \cap R^2 \cap T_a(p,n)$ such that the norm of x_n approaches infinity as n tends to infinity, and each x_n is not an element of C(m), for any m. This is possible since $V \cap R^2 \cap T_a(p,n)$ is an open subset of $T_a(p,n)$. Let $y \in \beta\{(x_n : n \geq 1\} - \{x_n : n \geq 1\} \subseteq \beta R^2 - R^2$. Since $x_n \in V$ for every n, $y \in \overline{V} - R^2$, and hence, $y \in \overline{W} \subseteq U$. Now define

$$f\colon \bigcup_{n\geq 1} C(n) \cup \{x_n : n \geq 1\} \to [0,1]$$

by $f(\bigcup_{n\geq 1} C(n)) = 0$, and $f(x_n) = 1$ for every n. Since $(\bigcup_{n\geq 1} C(n)) \cup \{x_n : n \geq 1\}$ is closed in R^2, there is a continuous extension of f to all of R^2, say $\overline{f}$. Then $\overline{f}$ can be continuously extended to βR^2, say by $\beta\overline{f}$. Now, $\beta\overline{f}(x_n) = f(x_n) = 1$ for every n implies $\beta\overline{f}(y) = 1$. Since

$$\beta\overline{f}(\bigcup_{n\geq 1} C(n)) = f(\bigcup_{n\geq 1} C(n)) = 0\ ,$$

it follows that $\beta\overline{f}(C) = 0$. Therefore, y is not an element of C, and hence, x is a limit point of $T_a^*(p,n) - C$.

By the above argument, $h(T_a^*(p,n)) \subseteq \overline{T}_b$ for every pair (p,n) with $p \in P_a$, $1 \leq n \leq p$. Therefore, $h(T_a) \subseteq \overline{T}_b$, which implies $h(\overline{T}_a) \subseteq \overline{T}_b$. Similarly, $h(\overline{T}_b) \subseteq \overline{T}_a$, and so $h(\overline{T}_a) = \overline{T}_b$.

Now, h must take the isolated components of $\overline{T}_a$ to the isolated components of $\overline{T}_b$. These are precisely the sets $T_a^*(p,n)$ and $T_b^*(q,m)$, respectively. So, for every pair (p,n) with $p \in P_a$, $1 \leq n \leq p$, we have $h(T_a^*(p,n)) = T_b^*(q,m)$ for some $q \in P_b$, $1 \leq m \leq q$.

Since $a \neq b$, either $P_a - P_b \neq \emptyset$ or $P_b - P_a \neq \emptyset$, so without loss of generality assume $P_b - P_a \neq \emptyset$, and let $q \in P_b - P_a$. For some pair (p,n), $p \in P_a$, $1 \leq n \leq p$, $h(T_a^*(p,n)) = T_b^*(q,1)$. We may assume $p < q$ since $p \neq q$. Then there are integers m,m' such that $1 \leq m \leq q$, $1 \leq m' \leq q$, with $h^{-1}(T_b^*(q,m)) = T_a^*(p,i)$ for some i, and $h^{-1}(T_b^*(q,m')) = T_a^*(p',n')$ for some $p' \in P_a$, $p' \neq p$, $1 \leq n' \leq p'$, and $|m - m'| = 1$. Now, $T_b^*(q,m)$ and $T_b^*(q,m')$ separate Y into two connected components and one disconnected component (since $|m - m'| = 1$). However, $h^{-1}(T_b^*(q,m)) = T_a^*(p,i)$ and $h^{-1}(T_b^*(q,m')) = T_a^*(p',n')$ separate X into three connected components, since $p \neq p'$. This is a contradiction; hence, X and Y are not homeomorphic.

So far, we have constructed c continua in $\beta R^2 - R^2$ no two of which are homeomorphic. We will now modify the construction to obtain 2^c nonhomeomorphic continua in $\beta R^2 - R^2$.

Let $S \subseteq A$ such that card $S = c$. There is a one-to-one correspondence between elements of S and real numbers r such that $0 \leq r < 2\pi$. So, each $a \in S$ corresponds to a unique $r_a \in [0,2\pi)$. Let $h_{r_a} : R^2 \to R^2$ be a rotation of R^2 by r_a radians. For each element, a, of S we will construct a continuum in the manner of the first section, except along the ray $h_{r_a}(\{(x,y) : x \geq 0, y = 0\})$. We will then take the union of these along with the Stone-Čech remainder of the set $\bigcup_{n\geq 1} \{(x,y) : x^2 + y^2 = n\}$. More precisely, let $R_a(p,n) = h_{r_a}(T_a(p,n))$, $p \in P_a$, $1 \leq n \leq p$, and $Q_a(n) = h_{r_a}(L(n))$. Then let R_S denote the union of the Stone-Čech remainders of the thickened rays $R_a(p,n)$, where $a \in S$, $p \in P_a$, $1 \leq n \leq p$; Q the union of the remainders of the rays $Q_a(n)$; and K the remainder of the union of the circles $\{(x,y) : x^2 + y^2 = n\}$, for $n \geq 1$. Let X be the closure in βR^2 of the union of these sets, i.e., $X = \overline{R_S} \cup \overline{Q} \cup K$. An argument similar to the one used in the first construction of the proof shows that X is a continuum. For another subset T of A such that $T \neq S$ and card $T = c$, we define R_T analogously, and let $Y = \overline{R_T} \cup \overline{Q} \cup K$. Then Y is also a continuum in $\beta R^2 - R^2$.

We will show that X and Y are not homeomorphic. Suppose h is a homeomorphism from X onto Y, and consider $\overline{R_S} \cup \overline{Q}$. Fix $a \in S$, and let N_1, N_2 be neighborhoods of the ray $h_{r_a}(\{(x,y) : x \geq 0, y = 0\})$ of radius 2,3, respectively. Let $f: R^2 \to [0,1]$ be a continuous function such that $f(N_1) = 0$ and $f(R^2 - N_2) = 1$. Then f has a continuous extension, βf, to all of βR^2. For $p \in P_a$, $1 \le n \le p$ and $m \ge 1$, since $R_a(p,n)$ and $Q_a(m)$ are contained in N_1, $\beta f(R_a^*(p,n))$ and $\beta f(Q_a^*(n))$ are both 0. On the other hand, if $a \ne a' \in S$, $q \in P_{a'}$, $1 \le n' \le q$, and $m' \ge 1$, then outside of some compact set (that depends on a'), $R_{a'}(q,n')$ and $Q_{a'}(m')$ are subsets of N_2. Therefore, $\beta f(R_{a'}^*(q,n'))$ and $\beta f(Q_{a'}^*(m'))$ are both 1. This implies that the closure of the union of all sets of the form $R_a^*(p,n)$ ($p \in P_a$, $1 \le n \le p$) and $Q_a^*(m)$ ($m \ge 1$) is isolated in $\overline{R}_S \cup \overline{Q}$. Hence, an argument identical to the one in the preceding section shows that $h(\overline{R}_S) = \overline{R}_T$.

Now, h must take the isolated components of $\overline{R}_S$ to the isolated components of $\overline{R}_T$. These are precisely the sets $R_a^*(p,n)$, $a \in S$, and $R_b^*(q,m)$, $b \in T$, respectively, by Lemma 3.3. So, for every $a \in S$ and pair (p,n) with $p \in P_a$, $1 \le n \le p$, we have $h(R_a^*(p,n)) = R_b^*(q,m)$, for some $b \in T$, $q \in P_b$, $1 \le m \le q$.

Either $S - T \ne \emptyset$ or $T - S \ne \emptyset$, so without loss of generality assume $T - S \ne \emptyset$, and let $b_0 \in T - S$. Let $q \in P_{b_0}$ and consider $R_{b_0}^*(q,1)$. For some $a_0 \in S$, $p \in P_{a_0}$, and $1 \le n \le p$, $h(R_{a_0}^*(p,n)) = R_{b_0}^*(q,1)$. Since $a_0 \ne b_0$, by an argument similar to the one used to show the continua in the first section were not homeomorphic, not every component of the form $R_{b_0}^*(q',m)$ can have as its inverse image under h a component of the form $R_{a_0}^*(p',n')$. Hence, there is an element a_1 of S, $p' \in P_{a_1}$, and $1 \leq n' \leq p'$, such that $a_1 \neq a_0$ and $h(R^*_{a_1}(p',n')) = R^*_{b_0}(q',m)$ for some $q' \in P_{b_0}$, $1 \leq m \leq q'$.

Now, $R^*_{a_0}(p,n)$ and $R^*_{a_1}(p',n')$ separate X into two connected components of $\overline{R}_S$. However, $h(R^*_{a_0}(p,n)) = R^*_{b_0}(q,1)$ and $h(R^*_{a_1}(p',n')) = R^*_{b_0}(q',m)$ separate Y into either one connected and one disconnected component (in case $q = q'$, $m = 2$), or into two connected components where one contains an infinite number of isolated components of $\overline{R}_T$ and the other contains only a finite number of isolated components of $\overline{R}_T$.

Since h is an onto homeomorphism that takes the isolated components of $\overline{R}_S$ to the isolated components of $\overline{R}_T$, this is a contradiction. Hence, X and Y are not homeomorphic.

Since A contains 2^c subsets of cardinality c, there are 2^c choices for X, no two of which are homeomorphic. Hence, by Lemma 3.5, since there are at most 2^c continua in $\beta R^2 - R^2$, there are exactly 2^c nonhomeomorphic continua in $\beta R^2 - R^2$.

Corollary 5.1: Let X and Y be as in the proof of the above theorem. Then there does not exist a continuous map f: X → Y that is a shape equivalence. In particular, X and Y are not homotopic.

Proof of Corollary 5.1

By Theorem 2.4, if $f(X) = Y$ is any continuous map which induces a shape equivalence, f is a homeomorphism. Hence, since X and Y are not homeomorphic, there does not exist such an f.

In the first part of the proof of Theorem 5.1, it would have been simpler to let A be the union of the regular and thickened rays, along with the curves C(n) and the positive x-axis, and let $X = \beta A - A \subseteq \beta R^2 - R^2$. However, in this case, any neighborhood of a point p in the remainder of the x-axis in X has dimension 2, by Lemma 3.4 (let $B_k = A \cap \{(x,y) \in R^2 : y \leq k\}$), yet p is not in $\overline{T}_a$. To see that p is not in $\overline{T}_a$, let $h: R^2 \to [0,1]$ where $h(\{(x,y) : x \geq 2, 0 \leq y \leq 1/(x^2)\}) = 1$, and $h(\{(x,y) : x \geq 2, y \geq 1/x\}) = 0$. Then $h(T_a) = 0$ implies $\beta h(\overline{T}_a) = 0$, but $\beta h(p) = 1$. Thus, if we had used the above definition for X instead of the one given in the proof of the theorem, we would not have been able to show that the sets $T_a^*(p,n)$ would be sent to the sets $T_b^*(q,m)$ under a homeomorphism from X onto Y.

CONCLUSION

In [8], [9], [10], [11], and Chapters Four and Five, the usefulness of shape theory in the study of the Stone-Čech compactification has been demonstrated, both in shape-theoretic results and results that are not specifically stated in shape-theoretic terms. From Chapter Four, we known that for $n \geq 3$, the Stone-Čech remainder of R^n contains at least c nonshape equivalent continua. It seems reasonable that the continua constructed in Theorem 4.3 may be shown to have different shapes, thereby exhibiting 2^c nonshape equivalent continua in $\beta R^n - R^n$ for $n \geq 3$. It is yet unknown whether or not two continua in the Stone-Čech remainder of R^n are shape equivalent if and only if they are homeomorphic (by Theorem 2.2 there cannot be a continuous function that induces such a shape equivalence unless it is a homeomorphism). If this were the case, then $\beta R^2 - R^2$ would also contain 2^c continua with different shapes by Theorem 5.1. We contrast these possibilities with the different shape types of continua in R^n.

For $n = 1$, the only continua in R^n are intervals, which have trivial shape. For $n \geq 3$, R^n contains exactly c continua with different shapes. This follows from the fact that there are c solenoids with nonisomorphic first Čech cohomology groups and, hence, different shapes and that R^n contains only c closed sets. For the case of R^2, however, it is not difficult to see that there are only countably many continua with different shapes, using the definition of shape as presented by S. Mardešić and J. Segal in [14]. The continua in R^2 can be classified up to shape according to the number of bounded components contained in the complement of each continuum. By taking neighborhoods of radius $1/k$ $(k \geq 1)$ of a given continuum X, we can obtain an inverse sequence with X as its limit that is homotopy equivalent to an inverse sequence of unions of circles; each bonding map will be the identity on some of the circles, sending the other circles to points. This follows from the fact that every $1/k$ neighborhood of X is homeomorphic to a disk with holes, which is homotopy equivalent to a union of tangent circles. Thus, X has the shape of a union of tangent circles, the number of circles corresponding to the number of bounded components, which will either be finite or countably infinite. By examining the first Čech cohomology groups of these continua, we see that two continua in R^2 are shape equivalent if and only if they have the same number of unbounded components in their complements. Hence, there are only countably many continua in R^2 with different shapes.

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BIOGRAPHICAL SKETCH

Alicia Winslow was born Alicia Marion Browner on June 20, 1954, in Ridgewood, New Jersey. She grew up in Mahwah, New Jersey, in an atmosphere of music and science. Deciding to pursue a career in mathematics, she entered the University of Florida in 1972 and received a bachelor's degree in mathematics in 1975. From 1975 to 1979, she has been a graduate student in the Department of Mathematics, teaching a variety of undergraduate courses.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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